BACK PAPER: ALGEBRA II

Date: **12th June 2019**
All vector spaces considered below are assumed to be finite dimensional.

(1) (20 points) Let V and W be vector spaces. Let ϕ and ψ be linear operators on V and W respectively. Consider the linear operator θ on $V \oplus W$ given by $\theta(v, w) = (\phi(v), \psi(w))$ for $v \in V$ and $w \in W$. Prove that the characteristic polynomial of θ is the product of the characteristic polynomials of ϕ and ψ. Give an example to show that the minimal polynomial of θ need not be the product of the minimal polynomials of ϕ and ψ.

(2) (20 points) Let A and B be two $n \times n$ matrices with entries in a field F such that the sum the rank of A and the rank of B is strictly less than n. Show that there exist a nonzero vector $x \in F^n$ such that $Ax = 0 = Bx$.

(3) (10+5+5 points) Prove or disprove using a counterexample the following statements.
   (a) Every square matrix with complex entries is conjugate to an upper-triangular matrix with complex entries.
   (b) Let $x(t)$ be a column vector of functions of length n and A be a $n \times n$ complex matrix. The differential equation $\frac{d}{dt}x(t) = Ax$ has a non zero solution.
   (c) Let $(V, \langle \cdot, \cdot \rangle)$ be a vector space together with a symmetric bilinear form. Let A be the matrix of the bilinear form with respect to some basis. If $\det(A) = 1$ then A is positive definite.

(4) (20 points) Prove that if A is an invertible matrix over complex numbers, then A^*A is Hermitian and positive definite. Define hermitian form and give an example of a hermitian form on $\mathbb{C}^2$ different from the standard hermitian form.

(5) (20 points) Let A be a real matrix that is normal and has real eigenvalues. Prove that A is symmetric.